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The graph of perfect matching polytope and an extreme problem

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ABSTRACT

For graph G, its perfect matching polytope $Poly(G)$ is the convex hull of incidence vectors of perfect matchings of G. The graph corresponding to the skeleton of $Poly(G)$ is called the perfect matching graph of G, and denoted by $PM(G)$. It is known that $PM(G)$ is either a hypercube or hamilton connected [D.J. Naddef, W.R. Pulleyblank, Hamiltonicity and combinatorial polyhedra, J. Combin. Theory Ser. B 31 (1981) 297–312; D.J. Naddef, W.R. Pulleyblank, Hamiltonicity in (0-1)-polytope, J. Combin. Theory Ser. B 37 (1984) 41–52]. In this paper, we give a sharp upper bound of the number of lines for the graphs G whose $PM(G)$ is bipartite in terms of sizes of elementary components of G and the order of G, respectively. Moreover, the corresponding extremal graphs are constructed.

1. Introduction

We follow [7] for the basic terminology. A *matching* of a graph G is a set M of lines such that each vertex of G is incident with at most one line in M. A matching M is *perfect* if each vertex of G is incident with exactly one line in M. A graph G is called *factorizable* if it admits a perfect matching. We call a graph G *factor-critical* if $G - v$ has a perfect matching for every vertex v of G. A line of graph G is called *allowed* if it lies in some perfect matching of G, and called *forbidden* otherwise. A line of G is said to be a *fixed* line if it belongs to all perfect matchings of G (called *allowed fixed* line) or no perfect matchings of G (called *forbidden* line). Other lines of G are said to be *unfixed* lines.

A factorizable graph G is *elementary* if its allowed lines form a connected subgraph. A connected factorizable graph G is said to be 1-*extendable* if all its lines are allowed. (Note that 1-extendable graphs are also called *matching covered* graphs in some literature). Clearly, every 1-extendable graph is elementary. Note that the converse does not hold in general.

The perfect matching polytope $Poly(G)$ of a graph G is the convex hull of the incidence vectors of the perfect matchings of G. From the results in [1], one can see that the graph corresponding to the skeleton of $Poly(G)$ can be directly defined as a simple graph in which the vertices are the perfect matchings of G, and two vertices are adjacent if and only if the symmetric difference of the corresponding two perfect matchings consists of exactly one alternating cycle of G. We call this graph to be the *perfect matching graph* of G, denoted by $PM(G)$.

In 1981 Naddef and Pulleyblank [8] introduced the concept of combinatorial polytope and showed that if the graph of a combinatorial polytope is bipartite then it is a hypercube and that if it is non-bipartite then it is hamilton connected. (A graph is hamilton connected if every pair of distinct vertices is joined by a hamiltonian path.) They showed that the same result holds for all (0, 1)-polytopes in [9] in 1984. Note that (0, 1)- polytopes include many well-known classes of polytopes, such as perfect matching polytope, matroid bases polytope, node packing or stable set (namely, independent set) polytope and permutation polytope.

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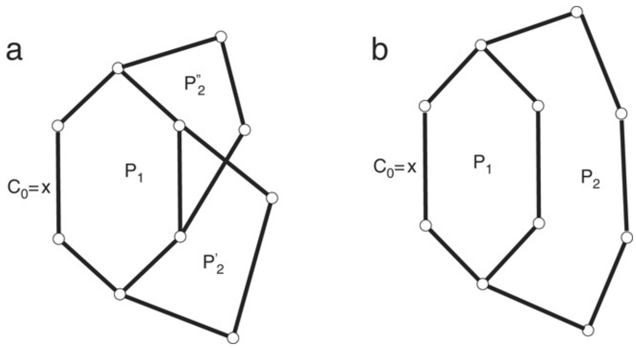

Fig. 1. C_2 is a 2-ear addition and C_2 is not a 2-ear addition.

In this paper, we give a sharp upper bound of the number of lines for the graphs G whose $PM(G)$ is bipartite in terms of sizes of elementary components of G and the order of G, respectively. Moreover, the corresponding extremal graphs are constructed.

2. Main results

We first recall the observation that for any factorizable non-bipartite graph G, after deleting all the forbidden lines of G, every component of the resulting graph is either an allowed fixed line of G (called a trivial elementary component of G) or a larger 1-extendable subgraph of G (called a non-trivial elementary component of G, the subgraph induced by the non-trivial elementary component in G is denoted by G_i). Theorem 2.9 in [8] implied the following

Lemma 2.1. *Let G be a factorizable non-bipartite graph. The perfect matching graph $PM(G)$ is bipartite if and only if every non-trivial elementary component of G is an even cycle.*

The proof of Lemma 2.1 needs the following definitions and some preliminary results.

Let X be a vertex set of G and $C_o(G-X)$ be the number of odd components of $G-X$. A set $X \subseteq V(G)$ is a *barrier* if $C_o(G-X) = |X| + def(G)$, where $def(G)$ is the deficiency of G, namely $def(G)$ is the number of vertices left uncovered by any maximum matching.

A subgraph H of G is said to be *nice* if $G - V(H)$ has a perfect matching. *An ear* of G is an odd path in G whose internal vertices (if there exist such vertices) have the degree of 2. According to the Two ear Theorem in [7], we know that every 1-extendable graph has a graded ear decomposition $(G_0, G_1, \dots, G_m = G)$ starting with an arbitrary line (or equivalently with an even cycle), in which each grade G_{i+1} results from the previous G_i through the addition of one or two ears, and G_i is a nice 1-extendable subgraph of G. In the case of a 1-ear addition the ear must join two different classes of $\mathscr{P}(G_i)$, whereas when 2-ear addition is employed, one ear joins two points of one class and the other joins two vertices of a different class of $\mathscr{P}(G_i)$, where $\mathscr{P}(G_i)$ is maximal barrier decomposition of 1-extendable graph G described in Lemma 2.4 and Theorem 2.5.

Theorem 2.2 (*[7]*). *Every non-bipartite* 1*-extendable graph G has a graded ear decomposition* $(G_0, G_1, G_2, G_3, \dots, G_m = G)$ *(where G_0 is a single line and G_1, an even cycle) such that either G_2 or G_3 is a 2-ear addition.*

Proof of Lemma 2.1. It is clear that the perfect matching graph of a non-bipartite graph G is the cartesian product of the perfect matching graphs of the non-trivial elementary components of G. The fact that the perfect matching graph of an even cycle is just K_2, implies the sufficiency of the theorem.

To show necessity, let G be a factorizable non-bipartite graph whose perfect matching graph $PM(G)$ is bipartite. By the result of [8] or [9], $PM(G)$ is a hypercube, namely, the cartesian product of K_2's. If we delete all the fixed lines of G, every non-trivial elementary component of the resulting graph is either a bipartite subgraph or a non-bipartite 1-extendable subgraph of G. For a non-trivial elementary component C of G, by Theorem 2.2, C has a graded ear decomposition $(C_0, C_1, C_2, C_3, \dots, C_m = C)$, where C_0 is a single line and C_1, an even cycle, such that either C_2 or C_3 is a 2-ear addition. Clearly, C_i is a nice 1-extendable subgraph of C. We claim that $m = 1$. Suppose, to the contrary, that $m \geq 2$. If C_2 is a 2-ear addition and $C_1 = C_0 + P_1$, $C_2 = C_0 + P_1 + \{P'_2 + P''_2\}$ (see Fig. 1(a)). It is easy to see that C_2 has exactly three perfect matchings M_1, M_2 and M_3. M_1 is the perfect matching of C_2 which contain the end-lines of P'_2 and P''_2, M_2, M_3 are two perfect matchings of C_2 which contain alternating cycle C_1. Since C_2 is a nice subgraph of C, there is a perfect matching F of $C - V(C_2)$. Then, $M_1 \cup F$, $M_2 \cup F$ and $M_3 \cup F$ are three perfect matchings of C such that the symmetric difference of each pair of them is an alternating cycle. Then, $M_1 \cup F$, $M_2 \cup F$ and $M_3 \cup F$ together with a perfect matching of the rest of elementary components of G will be three perfect matchings of G, which constitute a triangle of $PM(G)$. This contradicts the condition that $PM(G)$ is bipartite. If C_2 is not a 2-ear addition (see Fig. 1(b)), or the non-trivial elementary component of the resulting graph is a bipartite graph, the proof is analogous. This completes the proof. □

The concept of forcing lines was first defined in [3], which motivated by some chemical and physical concepts, see [5,6,10]. In Ref. [12] F. Zhang and X. Li characterize the hexagonal systems with forcing lines. H. Zhang and F. Zhang [13] give simple construction methods for several types of plane elementary bipartite graphs G that contain a forcing line. The *forcing number* of a perfect matching M of a graph G is the cardinality of the smallest subset of M that is contained in no other perfect matching of G [11]. In [4] the bounds of the forcing numbers of bipartite graphs are presented. In [11,2] the minimum forcing numbers for some special types of graphs such as benzenoid graph, torus and hypercube are determined. As a consequence of Lemma 2.1, we immediately have

Corollary 2.3. *Let G be a factorizable non-bipartite graph. If the perfect matching graph of G is bipartite, then all perfect matchings of G have the same forcing number m, and m equals the number of non-trivial elementary components of G.*

In order to consider the upper bound of the number of lines of non-bipartite graphs G whose $PM(G)$ is bipartite, we need to recall some known results.

Lemma 2.4 ([7]). *If G is elementary then the set of maximal barriers $\mathscr{P}(G) = \{S_1, S_2, \ldots, S_k\}$ is a partition of $V(G)$ (say canonical partition of $V(G)$).*

Theorem 2.5 ([7]). *Let $\mathscr{P}(G) = \{S_1, S_2, \ldots, S_k\}$ be the canonical partition of an elementary graph G. Then*

(a) *$X \subseteq V(G)$ is extreme in G if and only if $X \subseteq S_i$, for some i, $1 \le i \le k$.*
(b) *If x and y are vertices of G, then $G - x - y$ has a perfect matching if and only if x and y lie in different classes of $\mathscr{P}(G)$. In particular, a line $e = xy \in E(G)$ is allowed in G if and only if x and y lie in different classes of $\mathscr{P}(G)$.*
(c) *Let $S \subseteq V(G)$, then $S \in \mathscr{P}(G)$ if and only if $G - S$ has exactly $|S|$ components and each is factor-critical.*

A factorizable graph G is said to be *saturated* if $G + e$ has more perfect matchings than G for any line $e \in E(\overline{G})$. It is clear that a saturated graph must be connected. For any factorizable graph G, we can obtain a saturated graph by joining pairs of non-adjacent vertices as long as no new perfect matchings are produced [7]. Clearly, the non-bipartite graph whose perfect matching graph is bipartite with the maximum number of lines must be a saturated graph.

Theorem 2.6 ([7]). *If G is a saturated elementary graph, then the forbidden lines of G constitute point-disjoint complete subgraphs induced by the classes of $\mathscr{P}(G)$.*

Lemma 2.7 ([7]). *If G is a saturated elementary graph and $S \in \mathscr{P}(G)$, then $\mathscr{P}(G)$ must have at least $|S|$ singleton classes different from S.*

Now we suppose that G is a factorizable non-bipartite graph with maximal number of lines whose perfect matching graph is bipartite, we can characterize the non-trivial saturated elementary components of G.

Theorem 2.8. *Let G be a factorizable non-bipartite extremal graph whose perfect matching graph is bipartite, and G_i be a non-trivial saturated elementary component of G with order $2m_i$, then G_i is composed of an even cycle C_i with length $2m_i$ and a complete subgraph induced by m_i independent vertices of C_i.*

Before giving the proof of Theorem 2.8, we need the following definitions.

A hamiltonian cycle of a graph G is a cycle that contains every vertex of G. Let C be a hamiltonian cycle of graph G with a fixed cyclic orientation. For $x \in V(C)$, let x^+ and x^- denote the successor and predecessor of x respectively according to the given orientation of C. A line e, which joins a pair of non-adjacent vertices on C is a C-chord. Let $e = uv$ be a C-chord, a special crossing C-chord with respect to $e = uv$ is a C-chord which joins u^+ and v^+ (or u^- and v^-) according to the given orientation of C. It is clear that if there is a pair of special crossing chords on C, then there is a new hamiltonian cycle created by deleting uu^+ and vv^+ from C and adding the pair of special chords to C.

Proof of Theorem 2.8. By Lemma 2.1, every non-trivial elementary component of G is an even cycle. Let C_i be such an even cycle with length $2m_i$. We consider C_i be a bipartite graph, and color the vertices of C_i to be black and white. Clearly the end-vertices of every forbidden line have the same color. If the end-vertices of the forbidden lines are all the same color, say black (or white), then by Theorem 2.6, the subgraph induced by all the black (or white) vertices is a complete graph with the size of $\binom{m_i}{2}$, and there is no line among the white (or black) vertices. If the end-vertices of the forbidden lines are not the same color, we claim that the number of the forbidden lines is less than $\binom{m_i}{2}$. If the number of forbidden lines in G_i is equal to (or greater than) $\binom{m_i}{2}$, one can see that there is at least a pair of special crossing chords in C. Furthermore, the two special crossing chords are in a hamiltonian cycle other than C_i on the same vertex set of non-trivial elementary component of G, which contradicts the fact that the graph formed by all the allowed lines in G_i is an even cycle with length $2m_i$. Since G is an extremal graph, by Lemma 2.7, we note that G_i is a saturated elementary graph whose $\mathscr{P}(G_i)$ consist of S_i and $|S_i|$ singleton classes different S_i, where S_i is the set of all white (or black) vertices of C_i (see Fig. 2), and the graph induced by S_i is a complete graph. □

It is clear that S_i is the maximum barrier among all the canonical partition of $V(G_i)$.

Now we turn to the extreme problem. It is necessary to recall first the canonical construction procedure of saturated non-elementary graphs [7].

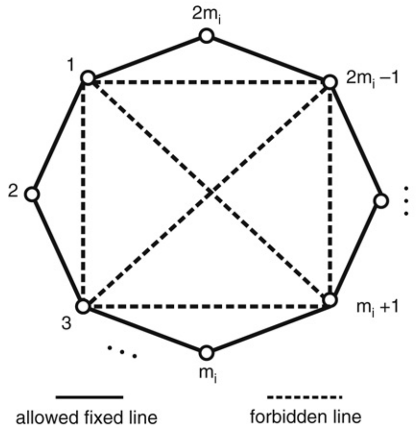

Fig. 2. A non-trivial saturated elementary component G_i of G.

Definition (*The Cathedral Construction*). Let $G_\circ$ be any saturated elementary graph. To each class $S \in \mathcal{P}(G_\circ)$, assign an (already constructed) saturated graph G_S or the empty set. For each $S \in \mathcal{P}(G_\circ)$ join every point of S to every point of G_S. In this case when G_S is not empty we will call the subgraph G_S the tower over S and S, the foundation of that tower. (Note that a tower and its foundation are point-disjoint).

Theorem 2.9 (*The Cathedral Theorem [7]*).

(a) *Every graph G built up by iterating the cathedral construction using smaller saturated graphs is itself saturated.*
(b) *The allowed lines of G are precisely those lines which are allowed in one of the elementary graphs used in one of the steps.*
(c) *Conversely, if G is any saturated graph, it can be built up using the cathedral construction starting with a saturated elementary graph $G_\circ$ and a collection of $|\mathcal{P}(G_\circ)|$ smaller saturated graphs (some perhaps empty) already constructed. The graph $G_\circ$ may be uniquely described as the subgraph of G induced by those points of G, which for each $x \in V(G)$ do not lie in $C(G-x)$.*

Note that the definition of $C(G-x)$ can be found in the Gallai–Edmonds decomposition of a graph G in [7], since we do not use it in the following, we omit it here. It is easy to verify that for any saturated graph G, the graph $G_\circ$ is exactly some saturated elementary component of G.

According to Theorem 2.9, the larger the order of a barrier in canonical partition of $V(G_i)$ is, the more the number of forbidden lines in G is. In the following theorem, we determine the sharp upper bound of the forbidden lines of a non-bipartite graph G whose perfect matching graph is bipartite, when the number of lines of elementary components of G are given. This is equivalent to determining the sharp upper bound of the lines of G.

Theorem 2.10. *Let G be a factorizable non-bipartite saturated graph whose perfect matching graph is bipartite. Suppose that G has t allowed fixed lines and n non-trivial saturated elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. Then the number of forbidden lines of G is no more than $f(t; m_1, m_2, \ldots, m_n) = \sum_{i=1}^{n}\binom{m_i}{2} + 2\sum_{1\le i<j\le n} m_i m_j + 2t\sum_{i=1}^{n} m_i + t(t-1)$.*

Proof. We will prove the theorem by induction on $n+t$. Clearly, the theorem is true for $n+t=1$. Assume that it holds for $n+t=k$. To prove it is true for $n+t=k+1$, we distinguish the following two cases.

Case (i). G has $t+1$ trivial elementary components and n non-trivial saturated elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. Deleting a trivial saturated elementary component e and the lines incident with it, we get a graph G' with fewer trivial saturated elementary components. By the induction hypothesis, G' has at most $f(t; m_1, m_2, \ldots, m_n)$ forbidden lines. We put back the line e to G' and add the forbidden lines as many as possible (see Fig. 3(a)). Then by the cathedral construction, the number of forbidden lines of G is at most $f(t; m_1, m_2, \ldots, m_n) + \sum_{i=1}^{n} 2m_i + 2t = f(t+1; m_1, m_2, \ldots, m_n)$.

Case (ii). G has t trivial saturated elementary components and $n+1$ non-trivial saturated elementary components $G_1, G_2, \ldots, G_{n+1}$ with orders $2m_1, 2m_2, \ldots, 2m_{n+1}$, respectively. Let us delete the non-trivial saturated elementary component G_{n+1} from G and the lines incident with it, we get a graph G' with fewer non-trivial saturated elementary components, by induction hypothesis, G' has at most $f(t; m_1, m_2, \ldots, m_n)$ forbidden lines. We put back the G_{n+1} and add the forbidden lines as many as possible (see Fig. 3(b)). Then by the cathedral construction, the number of forbidden lines of G is at most $f(t; m_1, m_2, \ldots, m_n) + m_{n+1}(2\sum_{i=1}^{n} m_i + 2t) + \binom{m_{n+1}}{2} = f(t; m_1, m_2, \ldots, m_{n+1})$. $\square$

Next, we define the set of non-bipartite graphs $\mu(t, n)$ inductively (according to the number of $n+t$) as follows:

(1) $\mu(1, 0)$ is the set of $\{K_2\}$, $\mu(0, 1)$ is the set of saturated elementary graphs which are composed of an even cycle C_i with length $2m_i$ and a complete subgraph induced by m_i independent vertices of C_i.

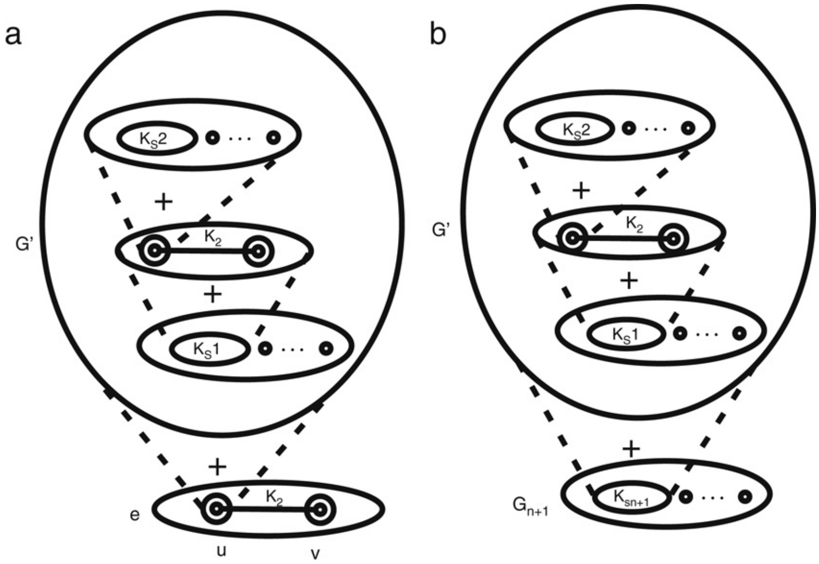

Fig. 3. The procedure of adding an allowed fixed line or a non-trivial saturated elementary component (some allowed lines between barriers are omitted).

(2) When the set $\mu(t, n)$ with $t + n = k$ has been defined.

(2.1) $\mu(t + 1, n)$ can be constructed in the following way: for any graph G in $\mu(t, n)$, taking a copy of $\{K_2\} = (u, v)$, by Theorem 2.6, $\mathcal{P}(K_2) = \{u, v\}$. According to Theorem 2.9, we assign the saturated graph G to $u(v)$, and join the vertex $u(v)$ to every vertex of G, assign an empty set to $v(u)$.

(2.2) $\mu(t, n+1)$ can be constructed in the following way: for any graph G in $\mu(t, n)$ and an isolated saturated elementary graph G_i belonging to $\mu(0, 1)$. According to Theorem 2.9, we assign the saturated graph G to S_i, and join every vertex of S_i to every vertex of G, for every singleton class of $\mathcal{P}(G_i)$ assign an empty set.

Note that all the graphs in $\mu(t, n)$ reach the upper bound $f(t; m_1, m_2, \ldots, m_n)$. Thus the upper bound is sharp. In fact, the graphs in $\mu(t, n)$ are just the extremal graphs reaching the sharp upper bound $f(t; m_1, m_2, \ldots, m_n)$. Furthermore, we have the following stronger result:

Theorem 2.11. *Let G be a saturated non-bipartite graph whose perfect matching graph is bipartite. Suppose that G has t allowed fixed lines and n non-trivial saturated elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. Then the number of forbidden lines of G is no more than $f(t; m_1, m_2, \ldots, m_n) = \sum_{i=1}^{n}\binom{m_i}{2}+2\sum_{1\le i<j\le n} m_im_j+2t\sum_{i=1}^{n} m_i+t(t-1)$. Furthermore, the upper bound is sharp and the set $\mu(t, n)$ is exactly the set of extremal graphs reaching the sharp upper bound.*

Proof. By the note before the statement of the theorem, we only need to show that any extremal graph G reaching the sharp upper bound $f(t; m_1, m_2, \ldots, m_n)$ belongs to $\mu(t, n)$. We prove it by the induction on $n + t$. Clearly, it is true for $n + t = 1$. Assume that it holds for $n + t = k$. By the inductive construction of the considered set, it is clear that to complete the proof by induction, we need distinguish the following two cases.

Case (i). G has t trivial saturated elementary components and $n + 1$ non-trivial saturated elementary components $G_1, G_2, \cdots, G_{n+1}$ with orders $2m_1, 2m_2, \ldots, 2m_{n+1}$, respectively. By deleting all vertices of the non-trivial saturated elementary component G_{n+1} and the lines incident with it, we get a graph G' with fewer non-trivial elementary components. We claim that G' belongs to $\mu(t, n)$. By the induction hypothesis, $\mu(t, n)$ is exactly the set of extremal graphs with the number of forbidden lines equal to $f(t; m_1, m_2, \ldots, m_n)$. If the claim is not true, the number of forbidden lines of G' is less than $f(t; m_1, m_2, \ldots, m_n)$. Now, by putting back the non-trivial saturated elementary component and adding the forbidden lines as many as possible, we get a graph with fewer forbidden lines. This contradicts that G is an extremal graph. Thus, our claim is true. Therefore, G can be constructed from G' by the step 2.2 in the procedure of construction of $\mu(t, n + 1)$.

Case (ii). G has $t + 1$ trivial saturated elementary components and n non-trivial saturated elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. The proof is analogous to Case (i), and so we omit it. □

Put the allowed lines of G together with the forbidden lines of G, we get

Corollary 2.12. *Let G be a saturated non-bipartite graph whose perfect matching graph is bipartite. Suppose that G has t allowed fixed lines and n non-trivial saturated elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. Then the number of lines of G is no more than $f'(t; m_1, m_2, \ldots, m_n) = \sum_{i=1}^{n}\binom{m_i}{2} + 2\sum_{1\le i<j\le n} m_im_j + (2t + 2)\sum_{i=1}^{n} m_i + t^2$. Furthermore, the upper bound is sharp.*

Now we consider another extremal problem. When the order of G is given, we can obtain a sharp upper bound on the number of lines of G in terms of the order of G.

Theorem 2.13. *Let G be a non-bipartite extremal graph (with respect to line) whose $PM(G)$ is bipartite. Then*

(i) *Every non-trivial elementary component of G is a cycle with length 4, and the number of trivial elementary component of G is at most one;*
(ii) *If $f(n)$ denotes the number of lines in G on n vertices, then*

$$f(n)=\begin{cases}\frac{1}{4}n^2+\frac{1}{4}n, & n=4k;\\ \frac{1}{4}n^2+\frac{1}{4}n-\frac{1}{2}, & n=4k+2.\end{cases}$$

Proof. (i) For an extremal graph G. Suppose that G has t trivial elementary components and n non-trivial elementary components $G_1, G_2, \ldots, G_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$ $m_i > 1$. By Lemma 2.1, the allowed lines in $G_1, G_2, \ldots, G_n$ are all cycles. We claim that there is no cycle with length greater than 4 among them. If the claim is not true, without loss of generality, we may assume that G_n has the length $4k$ ($k > 1$) or $4k+2$ ($k \geq 1$). We distinguish the following cases:

(1) If $2m_n = 4k$ ($k > 1$). By the constructed procedure in Theorem 2.11, we can construct a new graph G' by partitioning the cycle G_n into k's cycles with length 4. According to Corollary 2.12, we can compare the number of lines in G with that of G'.

$$\begin{aligned}
f'_G &= \sum_{i=1}^{n-1}\binom{m_i}{2}+\binom{2k}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+(2t+2)\sum_{i=1}^{n-1}m_i+(2t+2)2k+t^2\\
&= \sum_{i=1}^{n-1}\binom{m_i}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+(2t+2)\sum_{i=1}^{n-1}m_i+2k(2t+2)+(2k^2-k)+t^2\\
f'_{G'} &= \sum_{i=1}^{n-1}\binom{m_i}{2}+\sum_{i=1}^{k}\binom{2}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+\binom{k}{2}8+(2t+2)\sum_{i=1}^{n-1}m_i+(2t+2)2k+t^2\\
&= \sum_{i=1}^{n-1}\binom{m_i}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+(2t+2)\sum_{i=1}^{n-1}m_i+2k(2t+2)+(4k^2-3k)+t^2\\
f'_{G'}-f'_G &= (4k^2-3k)-(2k^2-k)=2k(k-1)>0
\end{aligned}$$

which contradicts the maximum of G.

(2) If $2m_n = 4k+2$ ($k \geq 1$). By the constructed procedure in Theorem 2.11, we can construct a new graph G' by partitioning the cycle G_n into k's cycles with length 4 and a trivial elementary component. In analogous manner, we have

$$\begin{aligned}
f'_G &= \sum_{i=1}^{n-1}\binom{m_i}{2}+\binom{2k+1}{2}+2\sum_{1\le i<j\le n-1}m_im_j+(4k+2)\sum_{i=1}^{n-1}m_i+(2t+2)\sum_{i=1}^{n-1}m_i+(2t+2)(2k+1)+t^2\\
&= \sum_{i=1}^{n-1}\binom{m_i}{2}+2\sum_{1\le i<j\le n-1}m_im_j+(4k+2)\sum_{i=1}^{n-1}m_i+(2t+2)\sum_{i=1}^{n-1}m_i+(2k^2+k)+t^2+(2t+2)(2k+1)\\
f'_{G'} &= \sum_{i=1}^{n-1}\binom{m_i}{2}+\sum_{i=1}^{k}\binom{2}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+\binom{k}{2}8+(2t+4)\sum_{i=1}^{n-1}m_i+(2t+4)2k+(t+1)^2\\
&= \sum_{i=1}^{n-1}\binom{m_i}{2}+2\sum_{1\le i<j\le n-1}m_im_j+4k\sum_{i=1}^{n-1}m_i+(2t+4)\sum_{i=1}^{n-1}m_i+2k(2t+4)+t^2+2t+1+(4k^2-3k)\\
f'_{G'}-f'_G &= (4k^2-3k)-(2k^2-k)=2k^2-1>0
\end{aligned}$$

which contradicts the maximum of G.

(3) We claim that the number of trivial elementary components is at most one.

Suppose, to the contrary, that there are two trivial elementary components, then by the constructed procedure in Theorem 2.11, we can construct a new graph G' in replacing the two elementary components by a cycle with length 4. According to Corollary 2.12, we immediately have

$$f'_G=\sum_{i=1}^{n}\binom{m_i}{2}+2\sum_{1\le i<j\le n}m_im_j+(2t+2)\sum_{i=1}^{n}m_i+t^2$$

$$f'_{G'} = \sum_{i=1}^{n} \binom{m_i}{2} + 1 + 2 \sum_{1\le i<j\le n} m_i m_j + (2t-2)\sum_{i=1}^{n} m_i + 4\sum_{i=1}^{n} m_i + (2t-2)2 + (t-2)^2$$
$$f'_{G'} - f'_G = 1 > 0$$

which contradicts the maximum of G. Combining (1), (2) with (3), we complete the proof of (i).

(ii) By using (i) and Corollary 2.12, we can obtain the desired result directly. □

In the case of bipartite graph, we can also consider the same kind of extremal problems, and obtain the following theorems. Since the proofs are similar, we omit them here.

Theorem 2.14. *Let G be a saturated bipartite graph whose perfect matching graph is bipartite. Suppose that G has t allowed fixed lines and n non-trivial elementary components $C_1, C_2, \ldots, C_n$ with orders $2m_1, 2m_2, \ldots, 2m_n$, respectively. Then the number of lines of G is no more than $f'(t; m_1, m_2, \ldots, m_n) = \sum_{1\le i<j\le n} m_i m_j + (t+2)\sum_{i=1}^{n} m_i + \frac{t(t+1)}{2}$. Furthermore, the upper bound is sharp.*

Theorem 2.15. *Let G be a bipartite extremal graph (with respect to line) whose $PM(G)$ is bipartite. Then*

(i) *Every non-trivial elementary component of G is a cycle with length 4, and the number of trivial elementary component of G is at most one;*
(ii) *If $f(n)$ denotes the number of lines in G on n vertices, then*

$$f(n) = \begin{cases} \frac{1}{8}n^2 + \frac{1}{2}n, & n = 4k; \\ \frac{1}{8}n^2 + \frac{1}{2}n - \frac{1}{2}, & n = 4k+2. \end{cases}$$

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